Statistical Modelling

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1. Linear Algebra and Calculus

We refresh some of the material in linear algebra and calculus needed for this course.

1.1 Linear Algebra Refresher

Recap 1.1 (Vector toolbox)**.** Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ be two vectors, then we recall the following definitions and properties:

- inner product: $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{a} = \sum_{i=1}^n a_i b_i$
- lenght: $\|\mathbf{a}\|^2 = \langle \mathbf{a}, \mathbf{a} \rangle = \sum_{i=1}^n a_i^2$
- Cauchy-Schwarz: $|\langle \mathbf{a}, \mathbf{b} \rangle|^2 \leq \langle \mathbf{a}, \mathbf{a} \rangle \cdot \langle \mathbf{b}, \mathbf{b} \rangle$
- orthogonality: $\mathbf{a} \perp \mathbf{b} \iff \langle \mathbf{a}, \mathbf{b} \rangle = 0$

Recap 1.2 (Orthonormal basis)**.** Let $\mathbf{y} = \sum_i b_i \mathbf{e}_i$ where the $\mathbf{e}_i$ form an orthonormal basis, i.e. $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{i,j}$. Then the coordinates can be calculated as $b_i = \langle \mathbf{y}, \mathbf{e}_i \rangle$.

Recap 1.3 (Matrix transpose and inverse)**.** Let $\mathbf{A} \in \mathbb{R}^{n \times p}$ and $\mathbf{B} \in \mathbb{R}^{p \times n}$ be two invertible matrices, then the following properties hold:

1. $(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$
2. $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
3. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
4. $\left(\mathbf{A}^{-1}\right)^\top = \left(\mathbf{A}^\top\right)^{-1} = \mathbf{A}^{-\top}$

A matrix $\mathbf{A}$ is called symmetric if $\mathbf{A} = \mathbf{A}^\top$.

Note that if $\mathbf{X}$ is a matrix, then $\mathbf{X}^\top \mathbf{X}$ and $\left(\mathbf{X}^\top \mathbf{X}\right)^{-1}$ (if it exists) are symmetric.

Recap 1.4 (Orthogonal projection)**.** Let $\mathbf{y} \in \mathbb{R}^n$ be a vector and S be a p-dimensional subspace spanned by linearly independent vectors $\mathbf{x}_1, \ldots \mathbf{x}_p$. If we define $\mathbf{X} = [\mathbf{x}_1 \cdots \mathbf{x}_p] \in \mathbb{R}^{n \times p}$, then the orthogonal projection matrix is $\mathbf{P}_S = \mathbf{X}\left(\mathbf{X}^\top \mathbf{X}\right)^{-1}\mathbf{X}^\top \in \mathbb{R}^{n \times n}$.

The orthogonal projection of $\mathbf{y}$ onto S in the basis of $\mathbb{R}^n$ is $\mathbf{y}_\perp = \mathbf{P}_S \mathbf{y}$ and we note the following properties:

1. $\mathbf{P}_S$ is symmetric, i.e $\mathbf{P}_S^\top = \mathbf{P}_S$
2. $\mathbf{P}_S$ is idempotent, i.e $\mathbf{P}_S^n = \mathbf{P}$

3. $\operatorname{tr} \mathbf{P}_S = p$
4. $\operatorname{rank} \mathbf{P}_S = p$
5. $\mathbf{y}_\perp = \arg\min_{\mathbf{s} \in S} \|\mathbf{y} - \mathbf{s}\|$
6. $\mathbf{1} - \mathbf{P}_S$ is also a projection

We also note that $\mathbf{P}_S(\mathbf{1} - \mathbf{P}_S) = \mathbf{0}$ and that the eigenvalues of $\mathbf{P}_S$ are in $\{0, 1\}$ with the geometric multiplicity of the eigenvalue 1 being p.

Recap 1.5 (Quadratic form, positive definite matrix)**.** Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix.

- $\mathbf{A}$ is called positive definite or p.d. if $\forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\} : \mathbf{x}^\top \mathbf{A} \mathbf{x} > 0$
- $\mathbf{A}$ is called positive semi-definite or p.s.d. if $\forall \mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{A} \mathbf{x} \geq 0$

Let λ_i denote the eigenvalues of $\mathbf{A}$. We note the following properties:

- if $\mathbf{A}$ is p.d., then $\lambda_i > 0$, $\operatorname{tr} \mathbf{A} > 0$, $\det\{\mathbf{A}\} > 0$

1.2 Multidimensional Calculus Refresher

Test